An extended actuarial model of rehabilitation versus new construction of housing

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Introduction

The actuarial model of rehabilitation versus new construction of public housing published recently in this journal (Johnstone, 1995) estimates the maximum justifiable expenditure on rehabilitation of public housing for each age at which rehabilitation takes place. This expenditure is expressed as a ratio of the costs to construct a new dwelling of similar size and quality. Johnstone's (1995) maximum cost ratio model extends Gleeson's (1992) prior model to take into account the depreciation of dwelling services with age. Johnstone's model estimates correct maximum cost ratios when annual maintenance costs over each age interval are proportional to the depreciation of dwelling services. This article extends and generalizes Johnstone's maximum cost ratio model to include regimes of annual maintenance costs which are not proportional to the depreciation of dwelling services.

Extended maximum cost ratio model

For the sake of brevity this article is written as an appendix to Johnstone (1995). Numbering of equations follows on from equation (23).

Depreciation of dwelling services is described by the function $D(h)$ where h is the effective age of a dwelling at the start of each age interval, the value of $D(h)$ is a proportion of the value of the dwelling services of a new dwelling in its first year of life ($0 \leq D(h) \leq 1$), and the units of $D(h)$ are dimensionless. If a dwelling undergoes full rehabilitation, then immediately following rehabilitation the effective age of the dwelling $h = 0$. The subsequent effective age $h = x - y$ where x is the actual age of dwellings at the start of an age interval and y is the age at which dwellings undergo full rehabilitation. The standard symbol x used in life tables for the age of human populations replaces the symbol k used in Johnstone (1995) for the actual age of dwellings.

Annual maintenance costs are assumed to be a function of the effective age of dwellings. The annual maintenance costs of a dwelling cohort of age x are estimated as the product of the function $M(h)$ and L_x which is the number of dwelling service years provided by a dwelling cohort over the age interval x to $(x + 1)$. The value of $M(h)$ is expressed as a proportion of the costs to construct

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a new dwelling ($0 \leq M(h) \leq 1$) and the units of $M(h)$ are in construction units per dwelling service year.

By definition, new construction does not undergo rehabilitation so $D(h) = D(x)$ and $M(h) = M(x)$. The benefit-cost ratio for new construction given by equation (13) in Johnstone (1995) is extended to include all possible regimes of annual maintenance as follows:

$$\frac{\text{Discounted Benefits}_{\text{NEW}}}{\text{Discounted Costs}_{\text{NEW}}} = \frac{\sum_{x=0}^{w} L_x \mathrm{D}(x) v^{x+1}}{\alpha l_0 + \sum_{x=0}^{w} L_x \mathrm{M}(x) v^{x}}. \tag{24}$$

The benefit-cost ratio for existing dwellings which undergo full rehabilitation given by equation (14) in Johnstone (1995) is extended likewise. The conversion factor for full rehabilitation (β) is replaced by the conversion factor (μ). The depreciation function $D(h) = D(x - y)$ and the maintenance function $M(h) = M(x - y)$ are effectively expressed by $D(x)$ and $M(x)$ instead owing to the structure of the equation:

$$\frac{\text{Discounted NBs}_{\text{REHAB}}}{\text{Discounted NCs}_{\text{REHAB}}} = \frac{v^y \frac{l_y}{l_0} \sum_{x=0}^{w} L_x \mathrm{D}(x) v^{x+1} - \sum_{x=y}^{w} L_x \mathrm{D}(x) v^{x+1}}{v^y \mu l_y + v^y \frac{l_y}{l_0} \sum_{x=0}^{w} L_x \mathrm{M}(x) v^{x} - \sum_{x=y}^{w} L_x \mathrm{M}(x) v^{x}} \tag{25}$$

In order to avoid overexpenditure on full rehabilitation, the benefit-cost ratio for full rehabilitation should be no less than that for new construction. A tacit assumption of this criterion is that the investment streams for both new construction and full rehabilitation can be increased in any proportion. From equations (24) and (25):

$$\frac{v^y \frac{l_y}{l_0} \sum_{x=0}^{w} L_x \mathrm{D}(x) v^{x+1} - \sum_{x=y}^{w} L_x \mathrm{D}(x) v^{x+1}}{v^y \mu l_y + v^y \frac{l_y}{l_0} \sum_{x=0}^{w} L_x \mathrm{M}(x) v^{x} - \sum_{x=y}^{w} L_x \mathrm{M}(x) v^{x}} \geq \frac{\sum_{x=0}^{w} L_x \mathrm{D}(x) v^{x+1}}{\alpha l_0 + \sum_{x=0}^{w} L_x \mathrm{M}(x) v^{x1}} \tag{26}$$

Equation (26) is inverted and simplified by multiplying each expression in the numerator and denominator of the left-hand side of the equation by $1/(v^y l_y)$ as follows:

$$\frac{\mu + \frac{1}{l_0} \sum_{x=0}^{w} L_x \mathrm{M}(x) v^{x} - \frac{1}{l_y} \sum_{x=y}^{w} L_x \mathrm{M}(x) v^{x-y}}{\frac{1}{l_0} \sum_{x=0}^{w} L_x \mathrm{D}(x) v^{x+1} - \frac{1}{l_y} \sum_{x=y}^{w} L_x \mathrm{D}(x) v^{x+1-y}} \leq \frac{\alpha l_0 + \sum_{x=0}^{w} L_x \mathrm{M}(x) v^{x}}{\sum_{x=0}^{w} L_x \mathrm{D}(x) v^{x+1}} \tag{27}$$

Rearrangement of equation (27) gives the extended maximum cost ratio μ at age x (μ_x) as follows:

$$\mu_x = \frac{\left[\alpha + \frac{1}{l_0}\sum_{x=0}^{w} L_x M(x) v^x\right]\left[\frac{1}{l_0}\sum_{x=0}^{w} L_x D(x) v^{x+1} - \frac{1}{l_y}\sum_{x=y}^{w} L_x D(x) v^{x+1-y}\right]}{\frac{1}{l_0}\sum_{x=0}^{w} L_x D(x) v^{x+1}} - \left[\frac{1}{l_0}\sum_{x=0}^{w} L_x M(x) v^x - \frac{1}{l_y}\sum_{x=y}^{w} L_x M(x) v^{x-y}\right] \quad (28)$$

The units of the extended maximum cost ratio (μ_x) are in construction units per dwelling which undergoes full rehabilitation. If "full rehabilitation" should take place at the age of zero, then the value of the extended maximum cost ratio $\mu_0 = 0$, as expected. Full rehabilitation at the age of zero represents the case of no rehabilitation. If annual maintenance costs are zero, then $M(x) = 0$ and equation (28) reduces to:

$$\mu_x = \frac{\alpha\left[\frac{1}{l_0}\sum_{x=0}^{w} L_x D(x) v^{x+1} - \frac{1}{l_y}\sum_{x=y}^{w} L_x D(x) v^{x+1-y}\right]}{\frac{1}{l_0}\sum_{x=0}^{w} L_x D(x) v^{x+1}} = \beta_x \quad (29)$$

as expected.

Annual maintenance costs can be non-zero and the maximum cost ratio $\mu_x = \beta_x$ if:

$$\frac{\left[\frac{1}{l_0}\sum_{x=0}^{w} L_x M(x) v^x\right]\left[\frac{1}{l_0}\sum_{x=0}^{w} L_x D(x) v^{x+1} - \frac{1}{l_y}\sum_{x=y}^{w} L_x D(x) v^{x+1-y}\right]}{\frac{1}{l_0}\sum_{x=0}^{w} L_x D(x) v^{x+1}} - \left[\frac{1}{l_0}\sum_{x=0}^{w} L_x M(x) v^x - \frac{1}{l_y}\sum_{x=y}^{w} L_x M(x) v^{x-y}\right] = 0 \quad (30)$$

or, after substitution and rearrangement,

$$\beta_x = \frac{\alpha\left[\frac{1}{l_0}\sum_{x=0}^{w} L_x M(x) v^x - \frac{1}{l_y}\sum_{x=y}^{w} L_x M(x) v^{x-y}\right]}{\frac{1}{l_0}\sum_{x=0}^{w} L_x M(x) v^x}. \quad (31)$$

The condition expressed by equation (31) is satisfied when $M(x) = kD(x)$ for each age x and some constant k. One example would be where annual maintenance costs are constant over the life span of dwellings and dwelling services do not undergo depreciation with age. Another example would be where annual maintenance costs decline at the same rate as the decline in value of dwellings services.

Data

This article uses the same data as used in Johnstone (1995) and additional maintenance data. The best available data on annual maintenance costs of New Zealand housing stock consist of records of New Zealand Housing Corporation dwellings (87,000 dwellings by 1990) which date back to the early 1940s. The writer estimates the average annual cost of maintaining a New Zealand dwelling to be 1.0 per cent of the cost of constructing a new dwelling. This estimate is based on a random sample of 25 dwellings located in Auckland. As records of maintenance fall well short of the 130 year economic life span of New Zealand housing stock, this article therefore investigates the impact of possible variations in maintenance costs with age on maximum cost ratios by assuming different regimes of maintenance costs based on the above estimate of maintenance costs.

Under maintenance regimes 1 and 2 annual maintenance costs are assumed to decline exponentially from 1.0 per cent of the cost of constructing a new dwelling at the age of zero to 0.5 per cent by the age of 130 years and to remain constant over the full economic life span of housing stock. Under maintenance regimes 3 and 4 annual maintenance costs are assumed to increase exponentially from 1.0 per cent of the cost of constructing a new dwelling at the age of zero to 2.0 and 4.0 per cent respectively by the age of 130 years.

Results and analysis

When the conditions that $M(x) = kD(x)$ for each age x are satisfied, the maximum cost ratio $\mu_x = \beta_x$. Under all other conditions the maximum cost ratio $\mu_x > \beta_x$ for each age $x > 0$. A sample of results are listed in Tables I, II, and III.

Higher expenditures on full rehabilitation are justified when increases in maintenance costs with age are greater. For example, Figure 1 shows that under scenario 4 ($r = 2$ per cent), diminishing value depreciation, and a discount rate of 12 per cent, $\mu_{40} = 0.36$ under maintenance regime 1 (1 to 0.5 per cent), $\mu_{40} = 0.37$ under maintenance regime 2 (constant), $\mu_{40} = 0.40$ under maintenance regime 3 (1 to 2 per cent), and $\mu_{40} = 0.43$ under maintenance regime 4 (1 to 4 per cent).

The maximum cost ratio β_x cannot exceed unity at any age x whereas the maximum cost ratio μ_x may exceed unity if increases in annual maintenance costs with age are substantial and benefits and costs are discounted using low

Age at rehabilitation	Maintenance regime: Exponentially decreasing 1.0 to 0.5%	Constant 1.00%	Exponentially increasing 1.0 to 2.0%	Exponentially increasing 1.0 to 4.0%
0	0.00	0.00	0.00	0.00
10	0.10	0.10	0.11	0.12
20	0.19	0.20	0.21	0.23
30	0.28	0.29	0.31	0.33
40	0.36	0.37	0.40	0.43
50	0.43	0.45	0.48	0.52
60	0.51	0.53	0.57	0.62
70	0.58	0.61	0.65	0.71
80	0.66	0.68	0.72	0.79
90	0.72	0.75	0.79	0.86
100	0.78	0.80	0.84	0.92
110	0.83	0.85	0.89	0.96
120	0.87	0.89	0.92	1.00
130	0.90	0.91	0.95	1.03

Table I.
Maximum cost ratios (diminishing value depreciation, scenario 4, discount rate = 12 per cent)

Age at rehabilitation	Maintenance regime: Exponentially decreasing 1.0 to 0.5%	Constant 1.00%	Exponentially increasing 1.0 to 2.0%	Exponentially increasing 1.0 to 4.0%
0	0.00	0.00	0.00	0.00
10	0.16	0.21	0.29	0.43
20	0.30	0.38	0.53	0.79
30	0.42	0.53	0.73	1.10
40	0.53	0.66	0.90	1.35
50	0.62	0.76	1.04	1.55
60	0.70	0.85	1.14	1.70
70	0.76	0.92	1.21	1.81
80	0.82	0.97	1.27	1.87
90	0.86	1.00	1.30	1.91
100	0.90	1.03	1.32	1.93
110	0.92	1.05	1.32	1.93
120	0.94	1.06	1.32	1.92
130	0.96	1.07	1.32	1.92

Table II.
Maximum cost ratios (diminishing value depreciation, scenario 1, discount rate = 0 per cent)

	Discount rate			
	0%	3%	6%	12%
0	0.00	0.00	0.00	0.00
10	0.25	0.16	0.13	0.11
20	0.47	0.30	0.24	0.21
30	0.64	0.43	0.35	0.30
40	0.78	0.56	0.46	0.39
50	0.89	0.67	0.56	0.48
60	0.97	0.76	0.65	0.56
70	1.03	0.84	0.73	0.63
80	1.06	0.91	0.81	0.70
90	1.08	0.96	0.87	0.77
100	1.09	0.99	0.91	0.83
110	1.09	1.02	0.95	0.87
120	1.09	1.03	0.98	0.91
130	1.08	1.04	1.00	0.94

Table III. Maximum cost ratios (scenario 3, diminishing value depreciation, maintenance regime 3)

discount rates. There are no guarantees that dwellings are able to undergo full rehabilitation within budget when the maximum cost ratio $\mu_x < 1$. Complete reversal of the depreciation of dwelling services is assured when $\mu_x \geq 1$ as an expenditure on full rehabilitation equal to the cost of constructing a new

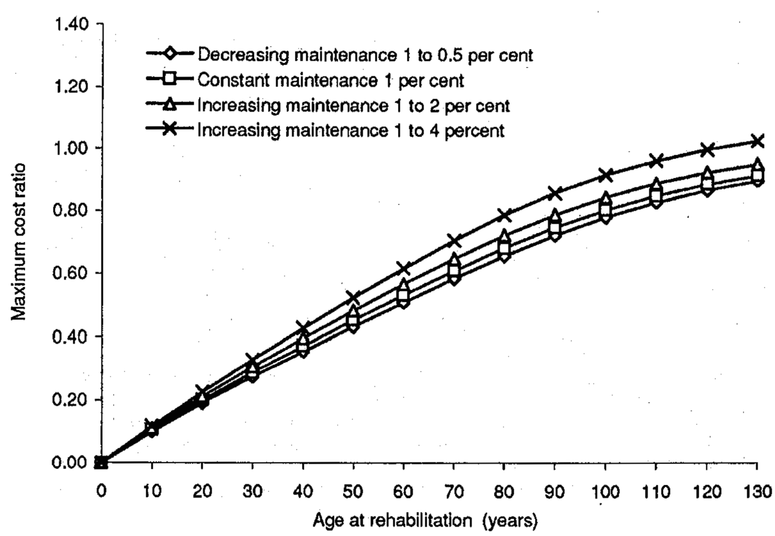

Figure 1. Maximum cost ratios (diminishing value depreciation, scenario 4, discount rate = 12 per cent)

dwelling can then be justified. If full rehabilitation cannot be undertaken within budget, then demolition and replacement is a viable alternative.

The age x at which the maximum cost ratio μ_x first exceeds unity occurs earlier when increases in annual maintenance costs with age are greater. For example, Figure 2 shows that under scenario 1 (r = 0 per cent) and discount rate of zero, μ_{90} = 1.00 under maintenance regime 2 (constant), μ_{50} = 1.04 under maintenance regime 3 (1 to 2 per cent), and μ_{30} = 1.10 under maintenance regime 4 (1 to 4 per cent).
Maximum cost ratios μ_x increase for each age x as the discount rate decreases. Under each maintenance regime, the age at which the maximum cost ratio μ_x may exceed unity therefore occurs earlier as the discount rate decreases. Low discount rates favour rehabilitation over new construction and may also justify demolition and replacement, especially when increases in annual maintenance costs with age are substantial. For example, Figure 3 shows that under scenario 3 (r = 1 per cent) and maintenance regime 3 (1 to 2 per cent) the maximum cost ratio μ_x first exceeds unity at the ages of 130, 110, and 70 years when the discount rates are 6, 3, and 0 per cent respectively.

Conclusion

This article has demonstrated the dynamics of maximum cost ratios under different maintenance regimes. Greater expenditure on the rehabilitation of public housing is justified when discount rates decline and earlier demolition and replacement of existing public housing may be justified as discount rates decline, especially should increases in annual maintenance costs with age be substantial. Because the model is based on assumed schedules of depreciation

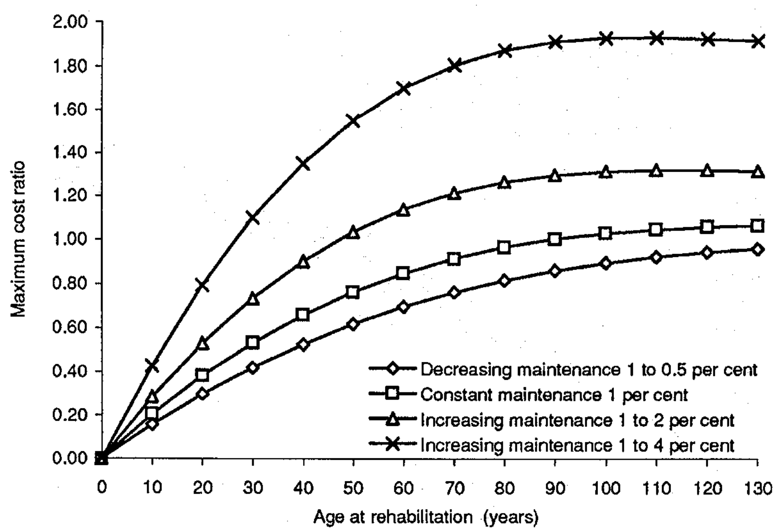

Figure 2.
Maximum cost ratios (diminishing value depreciation, scenario 1, discount rate = 0 per cent)

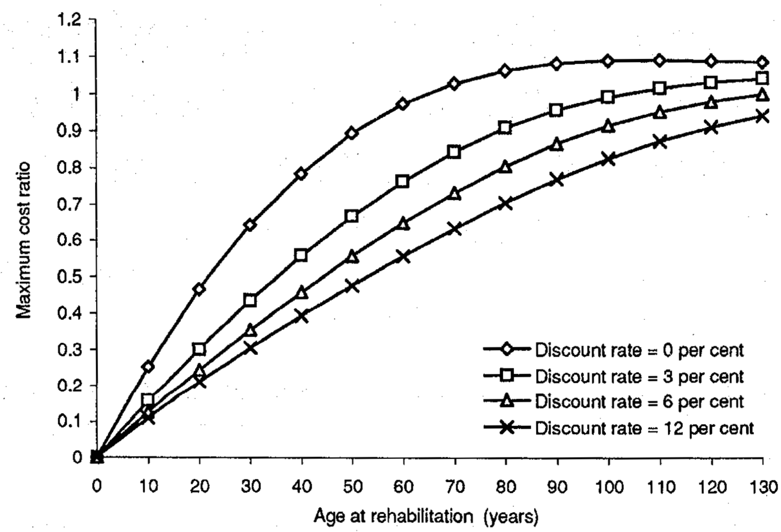

Figure 3.
Maximum cost ratios (scenario 3, diminishing value depreciation, maintenance regime 3)

and regimes of maintenance, longitudinal data on depreciation and maintenance over the full economic life span of housing are needed before the model can be fully validated.

References

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