An actuarial model of rehabilitation versus new construction of housing

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Introduction

Government and local authorities tend to provide public housing by investing in new construction and disregard or overlook the option of investing in the rehabilitation of existing dwellings. Investment in new construction does provide immediate housing, or dwelling services, but investment in rehabilitation also provides additional dwelling services by extending the remaining life expectancy of dwellings. The economic maxim that the use of resources should be optimized applies no less so than to housing stock. The resources embodied in housing stock are considerable, as illustrated by a recent perpetual inventory study of real capital stock in New Zealand. Dwellings formed over 23 per cent of the total value of the nation's capital stock of buildings, infrastructure, plant and equipment in 1989[1]. Housing policy should be framed and implemented so that investment in public housing provides the maximum possible dwelling services for the least cost. Investment would then be diverted into the rehabilitation of existing dwellings should the net benefit-cost ratio for rehabilitation exceed the benefit-cost ratio for new construction.

The costs of constructing new dwellings are easily estimated while the costs of fully rehabilitating existing dwellings must be done on a dwelling by dwelling basis. The benefits provided by rehabilitation or new construction are not so easily estimated. Underestimates of the benefits provided by rehabilitation based on unsubstantiated assumptions will lead to sub-optimal investment in new construction. Conversely, overestimates will lead to an over-investment in rehabilitation thus constituting a drain on national resources. An actuarial approach is required to estimate the benefits provided by rehabilitation and new construction for the same reasons that a medical examination of a room full of patients or a death notice in the newspaper do not provide reliable estimates of remaining or average life expectancies.

This article develops a model to estimate the maximum expenditure or budget which can be justified for rehabilitating existing dwellings for each age at which rehabilitation takes place. This budget is expressed as a ratio of the costs to construct a new dwelling of similar size and quality. If dwellings can be

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fully rehabilitated within budget, then investment in public housing should be channelled into rehabilitation. Conversely, if dwellings cannot be fully rehabilitated within budget, then investment in public housing should be channelled into new construction.

The maximum cost ratio model is based on the theories and life table models of classical population dynamics[2] and actuarial science[3]. Gleeson[4] has developed a prior model which is based on reliability theory[5]. Gleeson's estimates of maximum cost ratios are based on the assumption there is no decline in the value of dwelling services with age. In practice, the value of dwelling services does decline with age as a result of obsolescence, the causes of which include physical depreciation[6,7]. A number of empirical studies estimate the economic depreciation of dwellings and dwelling services, but none to date do so over the full economic life span of dwellings. This article, therefore, assumes straight line and diminishing value schedules of depreciation in order to establish the impact and dynamics of depreciation of dwelling services on maximum cost ratios.

Estimates of maximum cost ratios are based on estimates of mortality which pertains to probability of loss, survivorship, and life expectation. Variations and extensions of the theories and life table models of classical population dynamics have been developed over the past 200 years and actuaries, demographers, ecologists and engineers have applied these models to estimate the mortality of disparate populations[2,8]. However, it is only since the early 1980s that empirical studies of the mortality of housing stock have been carried out. In a pioneering study Gleeson adopted a life table modelling approach to estimate the mortality of a sample of Indianapolis housing stock[9]. In more recent years Johnstone[10] developed a simulation model based on the theories of classical population dynamics to estimate the mortality of New Zealand housing stock, and Komatsu *et al.*[11] adopted a model based on reliability theory to estimate the mortality of timber dwellings in Japan. The maximum cost ratios estimated in this article are based on Johnstone's estimates of the mortality of New Zealand housing stock as this is the best available mortality data for reasons addressed in the data section. A summary of life tables is located in the Appendix.

Maximum cost ratio model

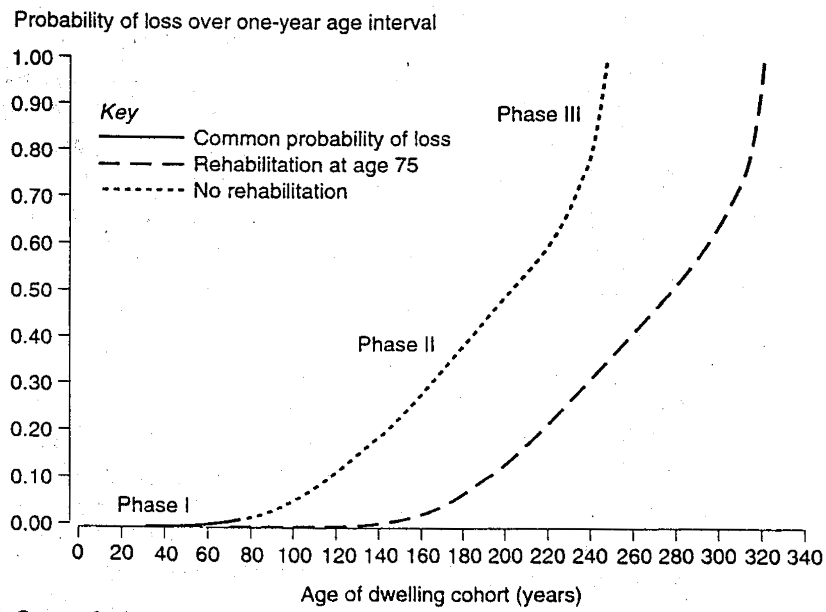
Assumptions

A number of assumptions are necessary in order to estimate maximum cost ratios. The implications, impact and justification for each of these assumptions are addressed as follows:

Assumption 1. Dwellings are assumed to undergo full rehabilitation to a level whereby the probability of loss of a fully rehabilitated dwelling over each successive age interval, subsequent to rehabilitation, is the same as that for a new dwelling subsequent to entry to the housing stock. After undergoing full rehabilitation the effective age of rehabilitated dwellings is zero and these dwellings are perceived as having the same desirability, and hence value (excluding land value), as that of a new dwelling of similar size and quality. This extreme assumption provides the means by which an absolute upper limit budget for full rehabilitation can be estimated.

Figure 1 shows a probability of loss schedule for new dwellings and dwellings which undergo full rehabilitation at the age of 75 years. A reversal in probability of loss is minimal when full rehabilitation takes place before the age of 75 years. Figure 2 graphs the corresponding surviving stock schedule where the entire surviving cohort undergoes full rehabilitation at the age of 75 years. The surviving stock schedule for fully rehabilitated dwellings initially follows the same schedule as that for new dwellings. In the process, a number of dwellings are lost over the time interval between entry to the housing stock and the age at which rehabilitation takes place. On full rehabilitation at the age of 75 years, the schedule then follows a scaled down schedule as for new dwellings. The end result is that the economic life span of the original dwelling cohort is extended to about 210 years.

Assumption 2. Dwellings are assumed to undergo full rehabilitation once only and the maximum cost ratio model is developed accordingly. Nonetheless, estimates of maximum cost ratios in this article apply equally for serial rehabilitation at regular intervals. This is because dwelling cohorts which have undergone recent rehabilitation can be regarded as being new cohorts which enter the housing stock with an effective age of zero. The number of times that



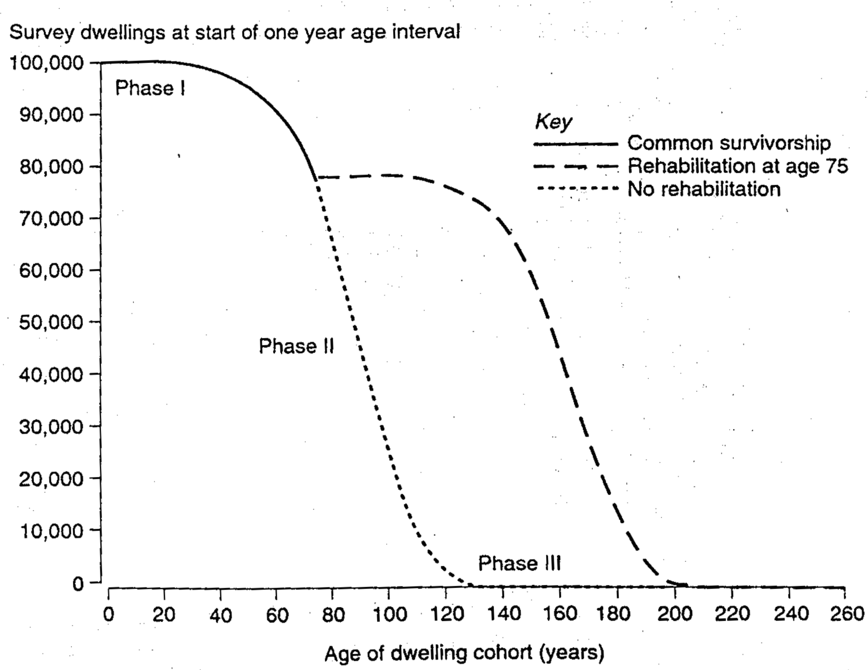
Source: [10]

Figure 1.
Probability of loss of dwellings rehabilitated at 75 years of age

serial rehabilitation of a dwelling can be undertaken is ultimately limited by the physical life span of the dwelling's structural system.

Assumption 3. For the sake of simplicity, dwellings are assumed to be homogeneous dwelling units. This assumption has minimal effect on estimated maximum cost ratios because the costs of rehabilitating a dwelling of a particular size and quality are in approximate proportion to the costs of constructing a new dwelling of similar size and quality.

Assumption 4. Maximum cost ratios should be based on empirical studies of the economic depreciation of dwelling services with age (rent or imputed rent, excluding rent for land) for the housing stock in question and not on assumed schedules of depreciation. However, no empirical study to date has been made of the economic depreciation of New Zealand dwellings or the economic decline in dwelling services with age. Two literature surveys of empirical studies of depreciation of dwellings – one by Malpezzi *et al.*[12] and the other by Baer[7] – do not provide satisfactory guidelines which can be applied with confidence to New Zealand housing stock. Not one study listed in the literature surveys estimates the depreciation of dwelling services or rent (excluding rent for land) over the full economic life span of dwellings. This article therefore assumes straight line and diminishing value depreciation of dwelling services. A straight line depreciation rate of 0.8 per cent per year is selected so that dwelling services depreciate to a value of zero by the age of 125 years, an age which approximates the economic life span of



Source: [10]

Figure 2.
Surviving dwellings rehabilitated at 75 years of age

New Zealand housing stock[10]. A diminishing value depreciation rate of 1.0 per cent per year is selected so that dwelling services decline to 49, 37 and 27 per cent of their original value at entry by the age of 70, 100 and 130 years respectively.

Assumption 5. For the sake of simplicity, the costs of rehabilitation and new dwelling construction are assumed to remain constant over time. The effect of this assumption on maximum cost ratios is minimized because these costs generally rise and fall simultaneously.

Assumption 6. The mortality of any housing stock is affected by a proportion of dwelling cohorts which have undergone full or partial rehabilitation. A negligibility assumption is therefore made. The empirically established regimes of mortality used within the maximum cost ratio model are assumed to approximate those regimes of mortality where no rehabilitation at all takes place.

Assumption 7. Previous levels and effectiveness of maintenance of the housing stock are assumed to be sustained in the future. This assumption is necessary because the probability of loss of dwellings is a function of a decline in value over time due to obsolescence, a decline which can be retarded, but not totally prevented, by appropriate levels of maintenance. Changes in the levels and effectiveness of maintenance of housing stock can therefore impact on the mortality of the housing stock.

Proxies for benefits and costs

The dwelling service years provided by new and rehabilitated dwellings serve as a proxy for benefits of rents or imputed rents (excluding rent for land). Dwelling construction units, the cost to construct one homogeneous dwelling, serve as a proxy for costs.

The numbers of new and rehabilitated dwellings are converted into dwelling construction units by multiplying by a conversion factor. The value of the unit conversion factor for new construction (α) is equal to one. The value of the unit conversion factor for full rehabilitation (β) is that proportion of the costs to construct a new dwelling required to fully rehabilitate a dwelling of a similar size and quality.

Description and setting up of models

Standard population dynamics notation is used, but standard terminology is replaced by dwelling related terminology. The main substitutes are "entries" in lieu of "births"; "surviving" in lieu of "living"; and "losses" or "departures" in lieu of "deaths". Although populations are exposed to hazards which can be described by a continuous function of mortality, departures in themselves are discrete events and data on mortality (age-specific dwelling losses and survivorship) are therefore also discrete. Life table models are constructed using discrete data and the standard description of a life table model uses summation symbols instead of integrals. This standard approach is used in this article. The age interval of the life table models of rehabilitation and new construction are set to be one year in order to simplify discounting of benefits and costs on an annual basis.

Benefit-cost ratio model of new construction (no depreciation)

The benefits, or dwelling service years provided by l_0 new dwelling entries over the economic life span w of the dwelling cohort, is given by:

$$\text{Benefits}_{\text{NEW}} = \sum_{k=0}^{w} L_k = T_0 \tag{1}$$

where k is the age of the dwelling cohort at the start of each age interval, L_k is the number of dwelling service years between the start of two adjacent age intervals k and $k + 1$, and T_0 is the total useful life provided by a dwelling cohort after the age of zero. It is assumed there is no decline in the value of dwelling services with age.

The costs of providing the above benefits are given by αl_0. The benefit-cost ratio is therefore given by:

$$\frac{\text{Benefits}_{\text{NEW}}}{\text{Costs}_{\text{NEW}}} = \frac{\sum_{k=0}^{w} L_k}{\alpha l_0} = \frac{T_0}{\alpha l_0} = \frac{e_0}{\alpha} \tag{2}$$

where e_0 is the average economic life on entry. As the unit conversion factor for new construction $\alpha = 1$, the above benefit-cost ratio has the same value as the average economic life on entry.

Benefits and costs are assumed to occur at the end and at the start of an age interval respectively and are discounted back to present values using the annual discount rate i and the discount factor v where:

$$v = \frac{1}{1+i}. \tag{3}$$

The discounted benefit-cost ratio for new construction is given by:

$$\frac{\text{Discounted benefits}_{\text{NEW}}}{\text{Discounted costs}_{\text{NEW}}} = \frac{\sum_{k=0}^{w} L_k v^{k+1}}{\alpha l_0}. \tag{4}$$

The benefit-cost ratio expressed above applies for all new construction at any point in time, including that construction which adds to the size of the housing stock. This is demonstrated as follows. Let a cohort of Z dwellings enter the housing stock at the beginning of year z. Because all dwellings are subject to the same regime of mortality, the benefits provided by Z dwellings are in the proportion Z/l_0 to the benefits provided by l_0 dwellings. The subsequent benefit-cost ratio is given by:

$$\frac{\text{Discounted benefits}}{\text{Discounted costs}} = \frac{v^z \sum_{k=0}^{w} \frac{Z}{l_0} L_k v^{k+1}}{v^z \alpha Z} = \frac{\sum_{k=0}^{w} L_k v^{k+1}}{\alpha l_0}. \tag{5}$$

Benefit-cost ratio model of full rehabilitation (no depreciation)

Let an entire surviving cohort of size l_y undergo full rehabilitation at the start of the age interval y to $(y + 1)$. The net benefits provided by l_y rehabilitated dwellings is equal to the sum of those benefits common to new dwellings from the age of 0 until y and a proportion l_y/l_0 of those benefits common to new dwellings from the effective age of 0 (the actual age is y) until the economic life span w, less those benefits from the age of 0 until w that would have been provided by l_0 new dwelling entries if full rehabilitation had not taken place. The net benefits are expressed as follows:

$$\text{Net benefits}_{\text{REHAB}} = \sum_{k=0}^{y-1} L_k + \sum_{k=0}^{w} \frac{l_y}{l_0} L_k - \sum_{k=0}^{w} L_k = \frac{l_y}{l_0} \sum_{k=0}^{w} L_k - \sum_{k=y}^{w} L_k \tag{6}$$

where k is the age of the dwelling cohort at the start of each age interval. If full rehabilitation occurs at the age $y = 0$, then $l_y = l_0$ and net benefits are zero, as expected.

The costs of providing the net benefits in equation (6) is given by βl_y. The net benefit-cost ratio is therefore expressed by:

$$\frac{\text{Net benefits}_{\text{REHAB}}}{\text{Costs}_{\text{REHAB}}} = \frac{\frac{l_y}{l_0} \sum_{k=0}^{w} L_k - \sum_{k=y}^{w} L_k}{\beta l_y}. \tag{7}$$

The above net benefit-cost ratio is the same, regardless of whether an entire surviving cohort, or a proportion only, undergoes full rehabilitation at the age y. This is demonstrated as follows. Let λ be the proportion of surviving dwellings l_y which are rehabilitated at age y, where $0 \le \lambda \le 1$. The net benefits provided by λl_y fully rehabilitated dwellings is equal to the sum of those benefits common to new dwellings from the age of 0 until y, a proportion $\lambda(l_y/l_0)$ of those benefits common to new buildings from the effective age of 0 until the economic life span w, and the remaining proportion $(1 - \lambda)$ of benefits provided by those new dwellings which do not undergo full rehabilitation, less those benefits that would have been provided by l_0 new dwelling entries from the age of 0 until w had full rehabilitation not taken place. The costs of providing these net benefits is $\lambda\beta l_y$. The resulting net benefit-cost ratio is expressed as follows:

$$\frac{\text{Net benefits}_{\text{PROPN REHAB}}}{\text{Costs}_{\text{PROPN REHAB}}} = \frac{\sum_{k=0}^{y-1} L_k + \sum_{k=0}^{w} \lambda \frac{l_y}{l_0} L_k + \sum_{k=y}^{w} (1-\lambda) \frac{l_y}{l_0} L_k - \sum_{k=0}^{w} L_k}{\lambda \beta l_y}$$

$$= \frac{\frac{l_y}{l_0} \sum_{k=0}^{w} L_k - \sum_{k=y}^{w} L_k}{\beta l_y}. \tag{8}$$

The positive and negative benefits expressed by the first and second terms in the numerator of equation (8) accrue at the end of the age interval y to $(y + 1)$ and successive age intervals. The discounted net benefit-cost ratio for full rehabilitation is therefore given by:

$$\frac{\text{Discounted net benefits}_{\text{REHAB}}}{\text{Disounted costs}_{\text{REHAB}}} = \frac{v^y \frac{l_y}{l_0} \sum_{k=0}^{w} L_k v^{k+1} - \sum_{k=y}^{w} L_k v^{k+1}}{v^y \beta l_y}. \tag{9}$$

Maximum cost ratio model (no depreciation)

In order to avoid over-expenditure on full rehabilitation, the discounted net benefit-cost ratio for rehabilitation should be no less than that for new dwellings. In other words:

$$\frac{v^y \frac{l_y}{l_0} \sum_{k=0}^{\infty} L_k v^{k+1} - \sum_{k=y}^{\infty} L_k v^{k+1}}{v^y \beta l_y} \geq \frac{\sum_{k=0}^{\infty} L_k v^{k+1}}{\alpha l_0}. \tag{10}$$

The maximum cost ratio of full rehabilitation to new construction under conditions whereby dwelling services undergo no depreciation with age (β_{ND}) is therefore expressed as follows:

$$\beta_{\text{ND}} = \frac{v^y \frac{l_y}{l_0} \sum_{k=0}^{w} L_k v^{k+1} - \sum_{k=y}^{w} L_k v^{k+1}}{v^y l_y} \cdot \frac{\alpha l_0}{\sum_{k=0}^{w} L_k v^{k+1}}$$

$$= \frac{\frac{1}{l_0} \sum_{k=0}^{w} L_k v^{k+1} - \frac{1}{l_y} \sum_{k=y}^{w} L_k v^{k+1-y}}{\frac{1}{l_0} \sum_{k=0}^{w} L_k v^{k+1}}. \tag{11}$$

The maximum cost ratio (β_{ND}) is a function of the age (or effective age in the case of serial rehabilitation) at which full rehabilitation takes place, probability of loss, and the selected discount rate. The expression of this function reduces to a ratio of discounted life expectancies, a result which is identical to that calculated by Gleeson[4].

Maximum cost ratio model (straight line depreciation)
The benefits of new dwellings at the end of each age interval are subject to the depreciation factor:

$$[1 - j(k+1)] \tag{12}$$

where j is the annual rate of straight line depreciation and k is the age at the start of each age interval.

From equation (4), the discounted benefit-cost ratio for new construction is given by:

$$\frac{\text{Discounted benefits}_{\text{NEW}}}{\text{Discounted costs}_{\text{NEW}}} = \frac{\sum_{k=0}^{w} L_k[1 - j(k+1)]v^{k+1}}{\alpha l_0}. \tag{13}$$

From equation (9), the discounted net benefit-cost ratio for full rehabilitation is given by:

$$\frac{\text{Discount NBs}_{\text{REHAB}}}{\text{Discount costs}_{\text{REHAB}}} = \frac{v^y \frac{l_y}{l_0} \sum_{k=0}^{w} L_k[1 - j(k+1)]v^{k+1} - \sum_{k=y}^{w} L_k[1 - j(k+1)]v^{k+1}}{v^y \beta l_y} \tag{14}$$

where rehabilitated dwellings are subject to the same depreciation schedule as for new dwellings.

From equations (10) and (11), the maximum cost ratio of full rehabilitation to new construction under conditions whereby dwelling services undergo straight line depreciation with age (β_{SL}) is expressed as follows:

$$\beta_{SL} = \frac{\frac{1}{l_0} \sum_{k=0}^{w} L_k[1 - j(k+1)]v^{k+1} - \frac{1}{l_y} \sum_{k=y}^{w} L_k[1 - j(k+1)]v^{k+1-y}}{\frac{1}{l_0} \sum_{k=0}^{w} L_k[1 - j(k+1)]v^{k+1}}. \tag{15}$$

The expression of the maximum cost ratio function (β_{SL}) also reduces to a ratio of depreciated and discounted life expectancies.

Maximum cost ratio model (diminishing value depreciation)
The benefits of new dwellings at the end of each age interval are subject to the depreciation factor:

$$z^{k+1} = \left(\frac{1}{1+s}\right)^{k+1} \tag{16}$$

where s is the annual rate of diminishing value depreciation and k is the age at the start of each age interval.

From equations (10) and (11) the maximum cost ratio of full rehabilitation to new construction under conditions whereby dwelling services undergo diminishing value depreciation with age (β_{DV}) is expressed as follows:

$$\beta_{DV} = \frac{\frac{1}{l_0}\sum_{k=0}^{w} L_k z^{k+1} v^{k+1} - \frac{1}{l_y}\sum_{k=y}^{w} L_k z^{k+1} v^{k+1-y}}{\frac{1}{l_0}\sum_{k=0}^{w} L_k z^{k+1} v^{k+1}}. \tag{17}$$

The existence of a maximum cost ratio is no guarantee that dwellings can be fully rehabilitated within budget. Nonetheless, for the moment assume that dwellings can be fully rehabilitated within budget and that at each age of rehabilitation an expenditure of the maximum justifiable budget is necessary to do so. The maximum cost ratio model is structured so that the net benefit-cost ratio for rehabilitation would be the same as the benefit-cost ratio for new construction for all ages at which rehabilitation takes place. The marginal benefit cost ratio of delaying rehabilitation at each age of rehabilitation – the ratio of the additional benefits gained to the additional cost required to gain those benefits by delaying rehabilitation by one time interval – would have the same value as the benefit-cost ratio for new construction.

Data

Maximum cost ratios are estimated using probability of loss data. These data should originate from empirical studies of mortality and not assumed regimes of mortality as to do so can lead to gross error[13]. The schedules of probability of loss data used within this article are based on Johnstone's[10,13] simulation model of the mortality of New Zealand housing stock. The simulation model was validated against New Zealand Department of Statistics data on the size of the housing stock at each quinquennial census since 1859, the distribution of the urban housing stock (75.3 per cent of the total housing stock) by age at August 1978 compiled by Valuation New Zealand, and New Zealand Department of Statistics data on annual new dwelling permits enumerated since 1921 for urban housing stock and since 1938 for rural housing stock.

The empirical studies carried out by Gleeson[9] and Komatsu *et al.*[11] make use of cross-sectional data which necessitates the adoption of the assumption that all dwelling cohorts have been, and will continue to be, exposed to the same regime of mortality. In other words, mortality is assumed to be static over time. Alternative hypotheses of mortality cannot be tested using cross-sectional data. By using a simulation model which is not restricted to assumptions of static mortality, Johnstone has established that New Zealand housing stock has been exposed to a dynamic regime of mortality between 1860-1980. Similar studies of other housing stocks may establish that these housing stocks are also exposed to dynamic, and not static, regimes of mortality.

The mortality of New Zealand housing stock is a function not only of age but also the expansion rate of the housing stock. The expansion rate will undoubtedly continue to fluctuate in the future, as it has done so in the past, so four simple scenarios of mortality only are considered in this article. Under scenarios 1, 2, 3, and 4 the housing stock is assumed to expand at a constant rate of 0, 0.5, 1 and 2 per cent per year respectively. Figure 3 graphs the probability of loss schedules to which dwelling cohorts would be subject under each scenario.

Different cases of rehabilitation within each scenario are examined. Dwellings undergo rehabilitation once only at the end of each decade of age ranging up to the age of 130 years. The case of "full rehabilitation" at the age of 0 is also included. Benefits and costs are discounted back to present values using the annual discount rates of 0, 3, 6 and 12 per cent. These discount rates have been selected purely to illustrate the impact of different discount rates on maximum cost ratios.

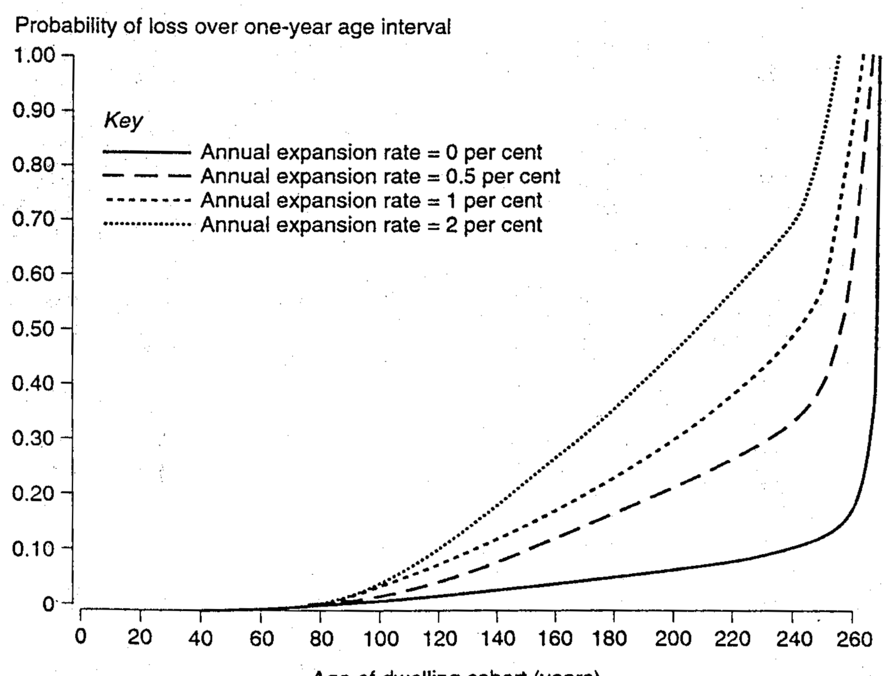


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Figure 3.
Probability of loss of dwellings under scenarios 1, 2, 3 and 4

Results

Maximum cost ratio: no depreciation of dwelling services with age

Figure 4 shows that the maximum cost ratios increase with the age at which full rehabilitation takes place and with increases in the expansion rate. The full results are listed in Table I.

A comparison of Figures 4 and 5 shows that the maximum cost ratios decrease with increases in the discount rate. Under conditions of no depreciation, market discount rates favour new construction. For example, a 50-year old dwelling cannot be fully rehabilitated within a budget of 4 per cent of the cost of a new dwelling of similar size and quality ($\beta_{50} = 0.04$ at a discount rate of 6 per cent).

The maximum cost ratios are sensitive to increases in the expansion rate at higher discount rates so that under conditions of no depreciation the use of maximum cost ratios would be impractical. For example, the maximum cost

Age at rehabilitation	Annual expansion rate r (%)				Annual expansion rate r (%)			
	0	0.5	1	2	0	0.5	1	2
	Discount rate = 0%				*Discount rate = 3%*			
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.08	0.10	0.10	0.11	0.01	0.02	0.03	0.03
20	0.15	0.19	0.21	0.22	0.03	0.05	0.06	0.07
30	0.23	0.29	0.31	0.33	0.05	0.09	0.10	0.13
40	0.31	0.38	0.41	0.44	0.07	0.14	0.16	0.19
50	0.38	0.47	0.50	0.54	0.11	0.20	0.23	0.27
60	0.45	0.56	0.59	0.63	0.15	0.27	0.31	0.37
70	0.52	0.64	0.67	0.71	0.20	0.34	0.40	0.46
80	0.59	0.71	0.74	0.78	0.25	0.43	0.49	0.56
90	0.65	0.76	0.80	0.83	0.31	0.51	0.57	0.64
100	0.70	0.81	0.84	0.87	0.37	0.58	0.64	0.71
110	0.74	0.85	0.88	0.91	0.43	0.64	0.71	0.77
120	0.78	088	0.90	0.93	0.48	0.70	0.76	0.82
130	0.81	0.90	0.93	0.96	0.53	0.74	0.80	0.86
	Discount rate = 6%				*Discount rate = 12%*			
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
10	0.00	0.00	0.01	0.01	0.00	0.00	0.00	0.00
20	0.01	0.01	0.02	0.02	0.00	0.00	0.00	0.00
30	0.01	0.03	0.04	0.05	0.00	0.01	0.01	0.01
40	0.02	0.05	0.06	0.08	0.00	0.01	0.02	0.03
50	0.04	0.08	0.11	0.14	0.01	0.03	0.04	0.05
60	0.06	0.13	0.16	0.21	0.02	0.05	0.07	0.09
70	0.09	0.19	0.24	0.30	0.03	0.08	0.11	0.15
80	0.12	0.26	0.32	0.39	0.05	0.13	0.17	0.23
90	0.16	0.33	0.40	0.49	0.07	0.18	0.24	0.31
100	0.21	0.41	0.49	0.57	0.10	0.25	0.31	0.40
110	0.26	0.48	0.56	0.65	0.14	0.31	0.39	0.48
120	0.31	0.55	0.63	0.71	0.17	0.38	0.46	0.55
130	0.36	0.61	0.68	0.76	0.21	0.44	0.52	0.62

Table I.
Maximum cost ratios β_{ND} (scenarios 1, 2, 3 and 4: no depreciation of dwelling services)

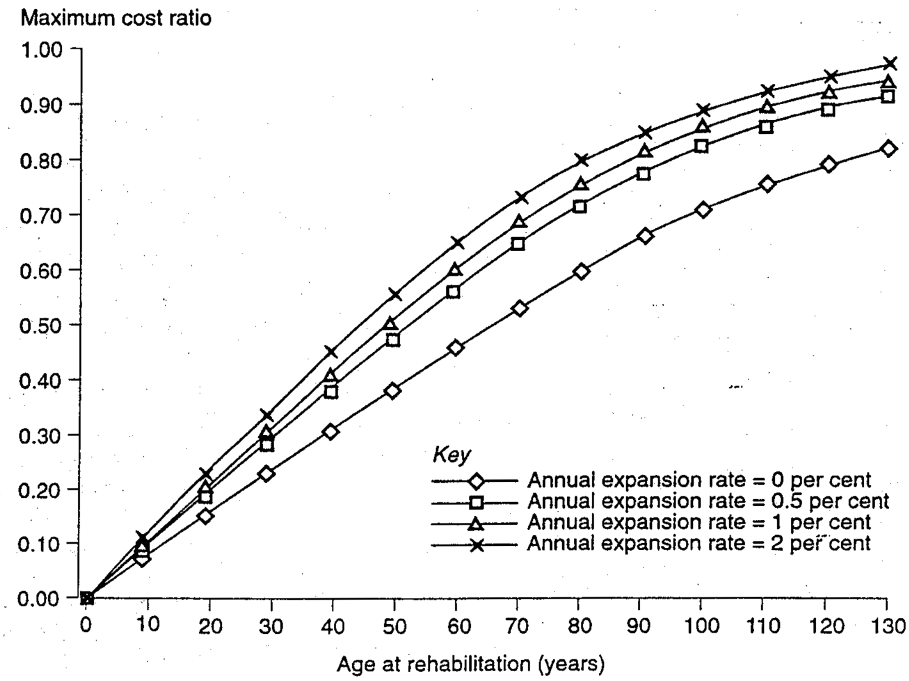

Figure 4. Maximum cost ratios β_{ND} (Discount rate = 0 per cent)

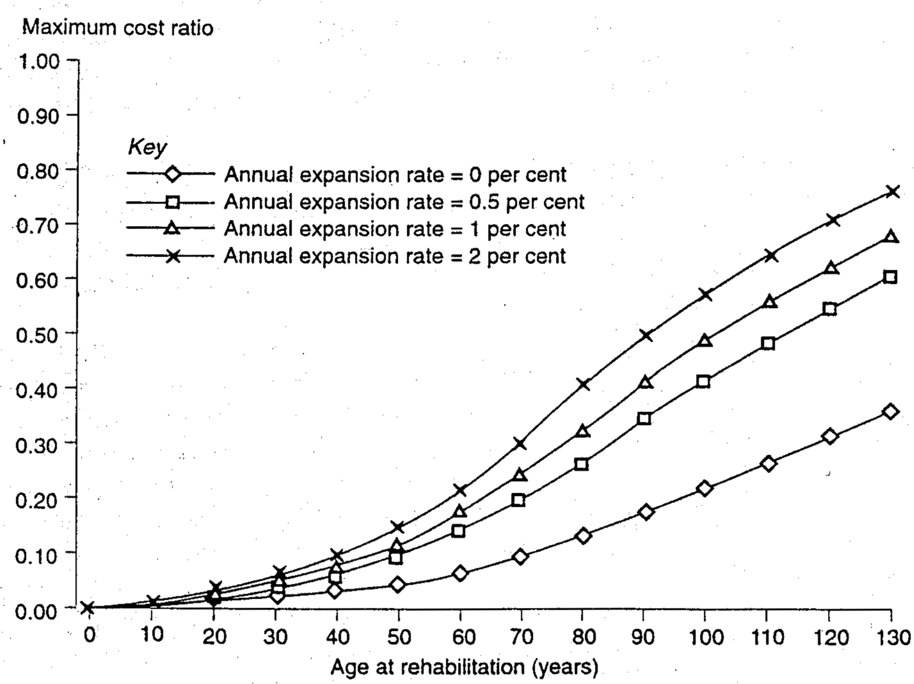

Figure 5. Maximum cost ratios β_{ND} (scenarios 1, 2, 3 and 4: discount rate = 6 per cent)

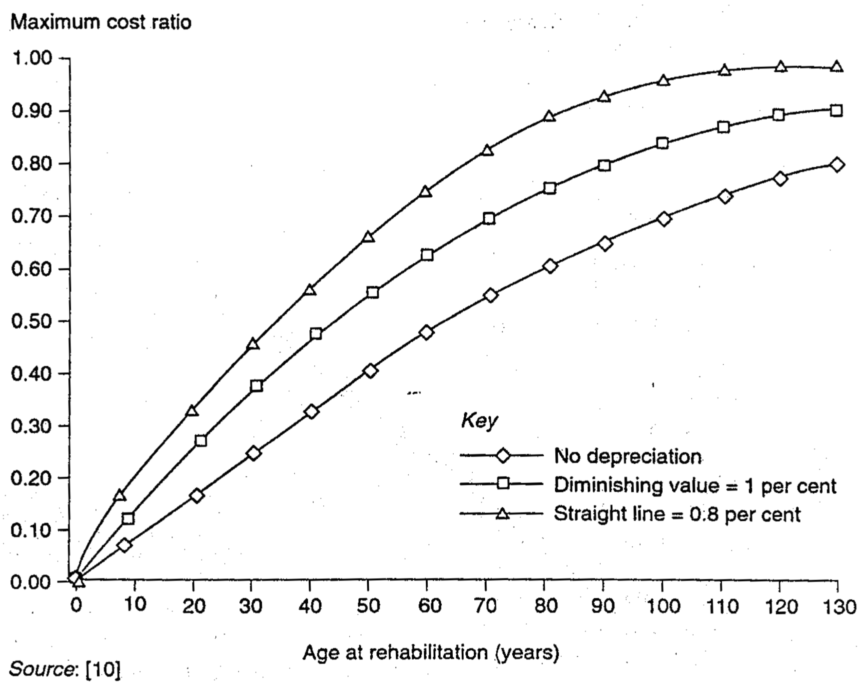

Figure 6. Maximum cost ratios β_{ND}, β_{SL}, β_{DV} (scenario 1, discount rate = 0 per cent)

ratio increases by 15 per cent from $\beta_{50} = 0.47$ under scenario 2 ($r = 0.5$ per cent) to $\beta_{50} = 0.54$ under scenario 4 ($r = 2$ per cent) when the selected discount rate is 0 per cent. In contrast, the maximum cost ratio increases by 75 per cent from $\beta_{50} = 0.08$ under scenario 2 ($r = 0.5$ per cent) to $\beta_{50} = 0.14$ under scenario 4 ($r = 2$ per cent) when the selected discount rate is 6 per cent per year.

Maximum cost ratio: depreciation of dwelling services with age

Figure 6 shows that greater maximum cost ratios are justified when the benefits of dwelling services undergo depreciation with age. For each age at which rehabilitation takes place $\beta_{SL} > \beta_{DV} > \beta_{ND}$. The full results are listed in Tables II and III.

Figure 7 shows that under conditions of straight line and diminishing value depreciation, compared with no depreciation, maximum cost ratios are relatively insensitive to increases in the discount rates. For example, when rehabilitation takes place at the age of 50 years and the discount rate is 12 per cent, $\beta_{SL} = 0.46$, $\beta_{DV} = 0.42$ and $\beta_{ND} = 0.05$ compared with $\beta_{SL} = 0.66$, $\beta_{DV} = 0.55$, and $\beta_{ND} = 0.38$ when the selected discount rate is 0 per cent.

Figures 8 and 9 show that the maximum cost ratios (β_{SL} and β_{DV} are insensitive to increases in the expansion rate when the discount rate is 12 per cent per year. For example, under straight line depreciation $\beta_{50} = 0.44$ when the expansion rate is 0 per cent per year compared with $\beta_{50} = 0.46$ when the

Age at rehabilitation	Discount rate = 0% Annual expansion rate r (%) 0	2	Discount rate = 12% Annual expansion rate r (%) 0	1	2
0	0.00	0.00	0.00	0.00	0.00
10	0.16	0.17	0.09	0.09	0.09
20	0.30	0.33	0.17	0.17	0.18
30	0.43	0.47	0.26	0.26	0.27
40	0.55	0.59	0.35	0.36	0.36
50	0.66	0.70	0.44	0.45	0.46
60	0.74	0.79	0.53	0.54	0.55
70	0.82	0.86	0.61	0.64	0.65
80	0.88	0.91	0.70	0.73	0.74
90	0.93	0.95	0.79	0.81	0.83
100	0.96	0.97	0.87	0.89	0.90
110	0.99	0.99	0.94	0.95	0.95
120	1.00	1.00	0.99	0.99	0.99
130	1.00	1.00	1.00	1.00	1.00

Table II. Maximum cost ratios β_{SL} (scenarios 1, 3 and 4 – depreciation of dwelling services)

expansion rate is 2 per cent per year. Under diminishing value depreciation β_{DV} is insensitive to increases in the expansion rate until the rehabilitation age of 40 years (β_{50} = 0.40 and β_{50} = 0.42 respectively), after which age the maximum cost ratio increases slightly with increases in the expansion rate. The use of maximum cost ratios would be practical under conditions of a fluctuating expansion rate.

Discussion

Maximum cost ratio models of the maximum justifiable expenditure on rehabilitation are based on the assumption that past regimes of mortality of

Age at rehabilitation	Discount rate = 0% Annual expansion rate r (%) 0	2	Discount rate = 12% Annual expansion rate r (%) 0	1	2
0	0.00	0.00	0.00	0.00	0.00
10	0.13	0.16	0.09	0.10	0.10
20	0.25	0.31	0.18	0.18	0.18
30	0.36	0.44	0.26	0.26	0.27
40	0.46	0.56	0.33	0.34	0.34
50	0.55	0.66	0.40	0.41	0.42
60	0.63	0.74	0.46	0.48	0.50
70	0.69	0.81	0.52	0.55	0.57
80	0.75	0.87	0.57	0.62	0.64
90	0.80	0.91	0.62	0.68	0.71
100	0.84	0.93	0.67	0.74	0.77
110	0.87	0.96	0.71	0.79	0.82
120	0.90	0.97	0.75	0.83	0.86
130	0.92	0.99	0.78	0.86	0.89

Table III. Maximum cost ratios β_{DV} (scenarios 1, 3 and 4: diminishing value depreciation of dwelling services)

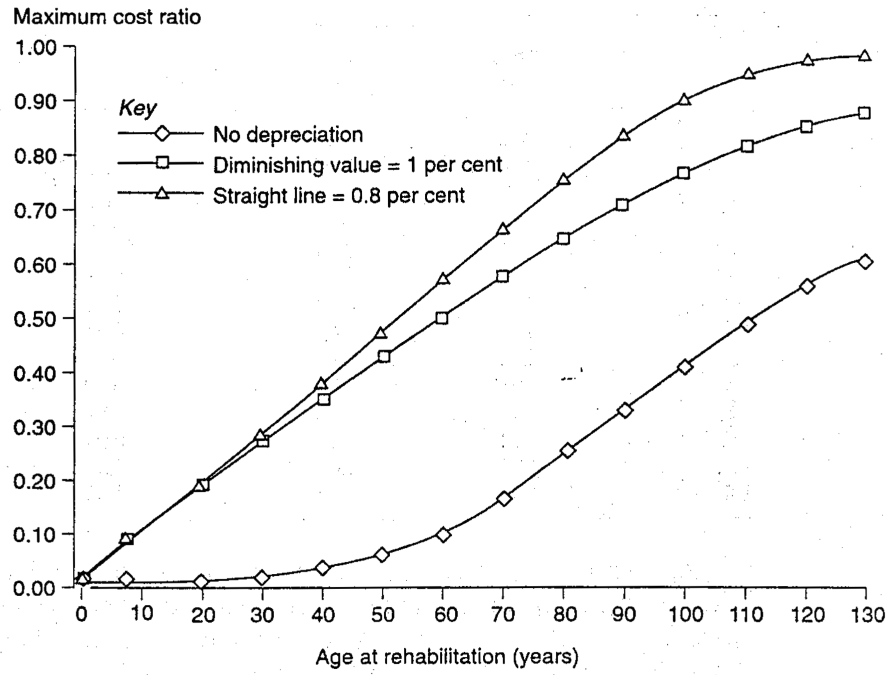

Figure 7.
Maximum cost ratios β_{ND}, β_{SL}, β_{DV} (scenario 4, discount rate = 12 per cent)

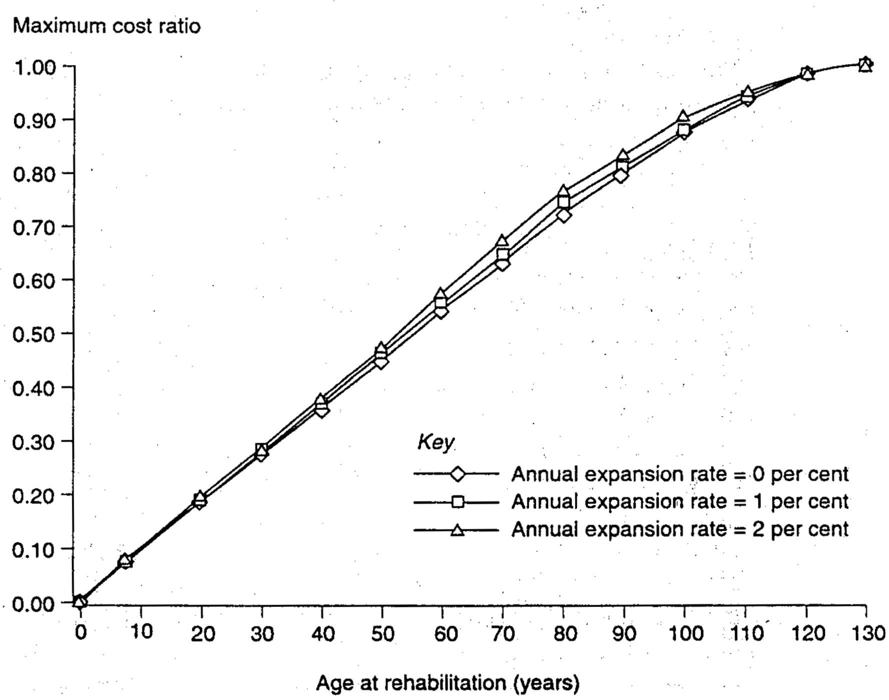

Figure 8.
Maximum cost ratios β_{SL} (discount rate = 12 per cent)

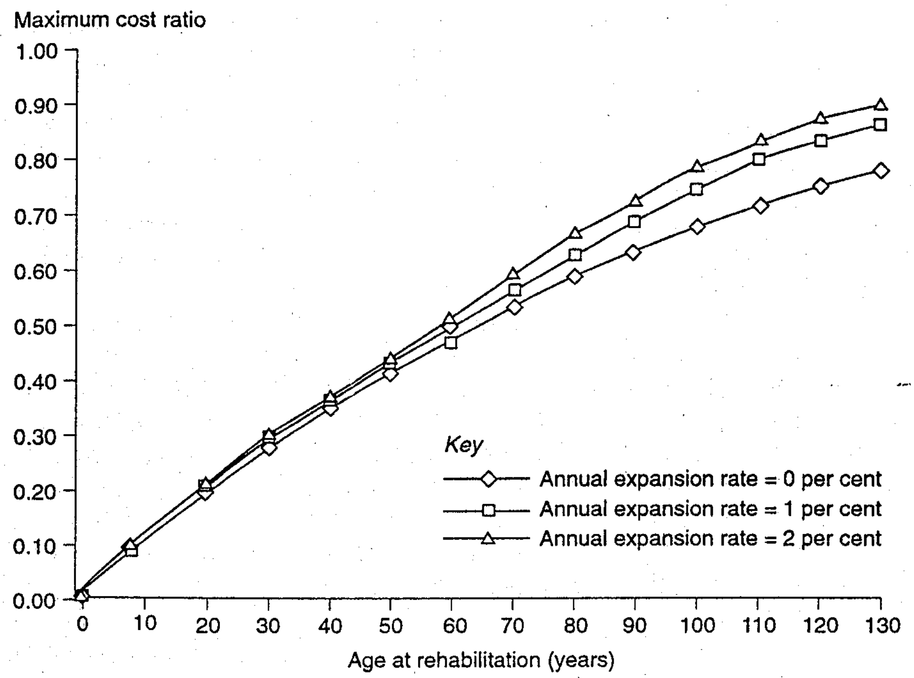

Figure 9. Maximum cost ratios β_{DV} (discount rate = 12 per cent)

housing stock and depreciation of dwelling services will continue in the future. A similar implicit assumption underpins all mortality life tables. The precision of the maximum cost ratio model is dependent on the realism of the schedules of mortality and depreciation which drive it. In this article the schedules of mortality are based on an empirical study of the mortality of New Zealand housing stock[10,14]. It should not be assumed that these schedules of mortality apply to housing stocks in other countries. A comparison of the economic life span of housing stocks gives an immediate gross indicator of whether housing stock life tables from one country may be applied to another. A distinguishing feature of the mortality of New Zealand housing stock is that it is dynamic and not static. Empirical studies of the mortality of housing stock based on cross-sectional data cannot establish whether the housing stock is subject to dynamic mortality or not as use of cross-sectional data requires the adoption of the implicit assumption that the mortality is static.

The schedules of depreciation in this article are based on assumed straightline and diminishing values of depreciation of dwelling services with age. These assumptions were necessary because empirical studies of the depreciation of dwelling services of New Zealand housing stock have yet to be carried out. Empirical studies listed in two recent literature surveys of the depreciation of dwellings do not provide satisfactory guidelines which can be applied with confidence to New Zealand housing stock[7,12]. This is because not one study listed in the literature surveys estimates the depreciation of dwelling services or rent (excluding rent for land) over the full economic life span of dwellings. The requirements of the maximum cost ratio model highlights the need for such studies. Empirical studies of the economic depreciation of dwellings over a full economic life span would need to be undertaken first. This is because estimates of the rate of decline in dwelling services with age are necessarily based on the correct proportioning of rent (or imputed rent) between the value of land and improvements, and valuations of improvements are based on estimates of replacement costs less estimates of depreciation.

The total costs of sustaining a nation's housing stock comprise the interdependent costs of new construction, replacement construction, rehabilitation and maintenance. By extending the life expectancy of housing stock, rehabilitation reduces the replacement rate which would be needed otherwise to provide a set quantity of dwelling services. A maximum cost ratio model does not take into account the impact of rehabilitation on the replacement rate and there may be an optimum timing of rehabilitation which minimizes the total costs to sustain a set quantity of dwelling services. If this should be so, then housing policy should promote and provide incentives for timely investment in rehabilitation.

A maximum cost ratio model forms a sub-model of a simulation model which can be developed to quantify and establish the dynamic relationship between rehabilitation, replacement, and maintenance within a housing stock. Such a simulation model would be of value not only for housing policy purposes, but also to the construction industry, which has an interest in forecasting dwelling construction. This is because as the expansion rate of a housing stock declines, replacement construction forms a greater proportion of total new construction. For example, should the expansion rate of New Zealand housing stock decline to 1 per cent per year then replacement construction would form about 40 per cent of total new construction[14]. Because future demand for replacement construction is determined largely by factors of obsolescence, an improved understanding of the obsolescence and depreciation of dwellings and a quantification and prediction of the resulting dynamics of rehabilitation, maintenance and replacement of housing stock would be invaluable.

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Appendix

A life table is a dynamic simulation model comprising a set of non-linear schedules which are constructed from age-specific loss and survivorship data. Each schedule within a life table is a mathematical transform of one another as follows[2,8].

Probability of loss (${}_nq_x$) is the proportion of dwellings that are standing at the beginning of an age interval which will be lost from the housing stock before reaching the end of the age interval. Probability of loss is given by:

$$ {}_nq_x = \frac{l_x - l_{x+n}}{l_x} \tag{18}$$

where l_x and l_{x+n} are the expected surviving stock which survive to the exact age x and $x + n$ marking the beginning of each respective age interval. By convention, the subscript for the age interval (n) is omitted when $n = 1$.

Stock losses (${}_nd_x$) gives the dwelling losses over successive age intervals x to $x + n$. Stock losses are given by:

$$ {}_nd_x = l_x - l_{x+n} \tag{19}$$

The stock in age interval (${}_nL_x$) gives the number of dwellings in each age interval within a stationary and stable housing stock. The stock in age interval can also be interpreted as the number of dwelling service years between the start of two adjacent age intervals x and $x + n$ provided by a single dwelling cohort. The stock in age interval is given by

$$ {}_nL_x = nl_x - {}_nd_x \; {}_na_x \tag{20}$$

where ${}_na_x$ is the average number of dwelling service years which are provided by dwellings lost over the age interval.

Total useful life (T_x) gives the number of stationary stock dwellings in an indicated age interval and all subsequent age intervals. Alternatively, total useful life is the total dwelling service years provided by a dwelling cohort after the age of x. Total useful life is given by:

$$T_x = {}_nL_x + {}_nL_{x+n} + {}_nL_{x+2n} + \dots = \sum_{k=0}^{\infty} {}_nL_{x+kn} \tag{21}$$

for $k = 0, 1, 2, \dots$

Life expectancy at age x (e_x), gives the average number of dwelling service years remaining to be provided by those dwellings which are still standing at the beginning of an age interval x and $x + n$. Life expectancy at age x, or remaining average economic life at age x, is given by

$$e_x = \frac{T_x}{l_x}. \tag{22}$$

The life expectancy on entry (e_0), or average economic life (on entry), is given by

$$e_0 = \frac{T_0}{l_0}. \tag{23}$$

The economic life span (w) of a housing stock is precisely defined as that age beyond which less than 0.1 per cent of an original dwelling cohort survives. Dwellings which no longer provide dwelling services are excluded.